J. Geom. *Online First*

DOI 10.1007/s00022-008-2015-2

Journal of Geometry

Generalizing the Equal Area Zones Property of the Sphere

Jeff Dodd and Vincent Coll

Abstract. We construct and study n-dimensional hypersurfaces in $(n+1)$-dimensional Euclidean space that satisfy a higher dimensional generalization of the equal area zones property of the sphere (that the surface area of a zone between two parallel planes depends only on the distance between the planes). These new hypersurfaces are hypersurfaces of revolution. The relative simplicity of their construction allows us to describe them in great detail, revealing some interesting curiosities and motivating further questions.

Mathematics Subject Classification (2000). 51N20.

Keywords. Equal area zones property; higher dimensional spheres; hypersurfaces of revolution.

1. Introduction

It is well known that the surface area of a zone sliced out of a sphere by two parallel planes depends only on the distance between the planes and not on the location of the zone, and is therefore proportional to that distance. This equal area zones property of the sphere dates back to Archimedes and is a standard calculus exercise (see [2]), and it makes an appearance in recreational mathematics in the solution of an old brain teaser about covering a disk with rectangles (see [1]). But it also has been the object of more serious study.

In 1951 Stamm [3] proved that this property characterizes the sphere among smooth closed convex surfaces. More precisely: if such a surface satisfies the property that the surface area of a region sliced from it by a pair of parallel planes is proportional to the distance between the planes, then the surface is a sphere. Moreover, this is the case even if the constant of proportionality is permitted to vary with the direction normal to the cutting planes, which we will call the *slicing direction.*

To the best of our knowledge, analogous properties for hypersurfaces in higher dimensional Euclidean spaces have not been investigated. Here we take a first step in this inquiry by looking among higher dimensional analogs of surfaces of revolution. We find that for each $n \geq 2$, there is just one smooth n-dimensional hypersurface of revolution in $n+1$ dimensional Euclidean space that satisfies a generalized equal area zones property. We call this hypersurface the equizonal n-ovaloid, EO^n. (Curiously, it is a C^∞ manifold when n is even, but only $C^{(n-1)/2}$ when n is odd.)

For $n = 2$, the equizonal n-ovaloid is an ordinary sphere, but for $n > 2$ it is not and it satisfies a generalized equal area zones property in only one slicing direction. So the fact that the two dimensional sphere satisfies the equal area zones property in all slicing directions is a unique feature of three dimensional Euclidean space. We mensurate the equizonal n-ovaloids completely and find that, in some respects, they more nearly satisfy higher dimensional analogs of the properties of the sphere derived in Archimedes' *On the Sphere and the Cylinder* than do n-dimensional spheres. Finally, we pose some questions for further study.

2. Construction of the equizonal ovaloids

We define a *hypersurface of revolution* to be an n-dimensional hypersurface imbedded in $\mathbb{R}^{n+1}$ whose cross sections orthogonal to a particular line, which we label the x-axis, are $(n-1)$-dimensional spheres centered on the x-axis, as illustrated in Figure 1. There the notation $S^{n-1}(f(x))$ indicates an $(n-1)$-dimensional sphere of radius r, where r is a function $f(x)$ that we call a *profile function*. We wish to determine which such hypersurfaces satisfy the *generalized equal area zones property* that the n-volume sliced from the hypersurface by two hyperplanes $x = a$ and $x = b$ depends only on the distance between the hyperplanes, $b - a$.

The $(n-1)$-volume of $S^{n-1}(r)$ is given by

$$\mathrm{vol}\big[S^{n-1}(r)\big] = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1}\,, \quad n \geq 2$$

where Γ denotes the gamma function

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\, \mathrm{d}t\,, \quad \mathrm{Re}(z) > 0\,.$$

So the n-volume of a slice of the hypersurface between $x = a$ and $x = b$ is given by

$$S = \int_a^b \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \big[f(x)\big]^{n-1} \sqrt{1 + \big[f'(x)\big]^2}\, \mathrm{d}x\,. \tag{1}$$

If S depends only on $b - a$, and not on the location of the slice, then the integrand must be a positive constant. That is, for some $c > 0$,

$$\big[f(x)\big]^{n-1} \sqrt{1 + \big[f'(x)\big]^2} = c\,.$$

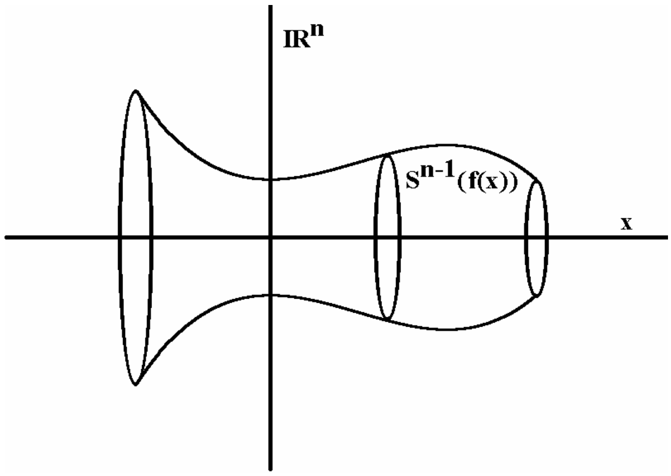

FIGURE 1. Schematic representation of an n-dimensional hypersurface of revolution.

Rescaling the coordinates by $x \to x/C$ and $f \to f/C$ where $C = c^{1/(n-1)}$ yields the same equation but with $c = 1$, so C merely represents a linear scaling factor. Taking $c = 1$ and squaring both sides of the equation yields:

$$\big[f(x)\big]^{2n-2} + \big[f(x)\big]^{2n-4}\big[f(x)f'(x)\big]^2 = 1\,. \tag{2}$$

The change of variables $v = f^2$ reduces this to a separable equation which can be solved by quadrature. The solutions are $f(x) = 1$ (representing a *hypercylinder* of radius 1) and

$$x = \frac{1}{2}\int_0^{[f(x)]^2} \frac{t^{\frac{n-2}{2}}}{\sqrt{1-t^{n-1}}}\,\mathrm{d}t + d\,, \quad 0 \le f(x) \le 1 \tag{3}$$

where d is a constant representing a horizontal shift along the x-axis, so that we may as well take $d = 0$.

We call the n-dimensional hypersurface of revolution generated by the function f defined by (3) the *unit equizonal n-hemisphere*, since its largest $(n-1)$-dimensional spherical cross section has radius 1.

Compact hypersurfaces of revolution of C^1 smoothness satisfying the generalized equal area zones property in the x-direction can be obtained by capping an n-dimensional hypercylinder of radius 1, and of any length, on each end with a unit equizonal n-hemisphere. In the special case where the hypercylinder has length zero, we call the resulting hypersurface the *unit equizonal n-ovaloid*, and denote it $EO^n(1)$; see Figure 2 where

$$x_{\max} = f^{-1}(1) = \frac{1}{2}\int_0^1 \frac{t^{\frac{n-2}{2}}}{\sqrt{1-t^{n-1}}}\,\mathrm{d}t\,.$$

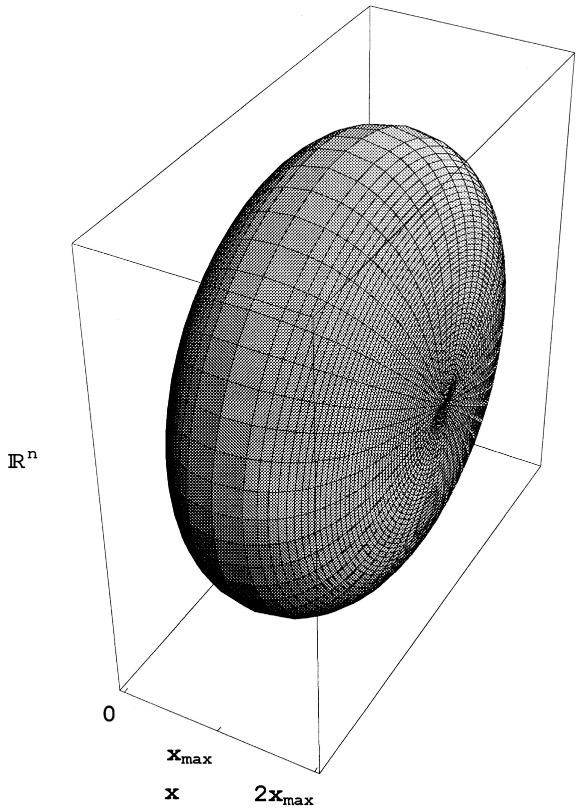

FIGURE 2. A 2-dimensional representation of a unit equizonal n-ovaloid $EO^n(1)$.

(Note that as n increases, $EO^n(1)$ becomes increasingly oblate. Because the $(n-1)$-volume of its cross section $S^{n-1}(f(x))$ increases in proportion to $[f(x)]^{n-1}$, the graph of the profile function f must level off more and more rapidly as n increases in order to maintain the generalized equal area zones property.)

Scaling the linear dimensions of $EO^n(1)$ by a factor of r produces a 1-parameter family of equizonal n-ovaloids which we denote $EO^n(r)$. Surprisingly, the smoothness of an equizonal n-ovaloid depends on its dimension n.

Proposition 1. *For even $n > 2$, the only n-dimensional compact C^∞ hypersurfaces of revolution which satisfy the generalized equal area zones property are the equizonal n-ovaloids. For odd $n > 2$, there are no n-dimensional compact C^∞ hypersurfaces of revolution which satisfy the generalized equal area zones property,* *and the smoothest such compact hypersurfaces of revolution are the equizonal n-ovaloids which are $C^{(n-1)/2}$.*

Proof. Inserting the Maclaurin series

$$\frac{1}{2}\frac{1}{\sqrt{1-u}} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \ldots (2k-1)}{2^{k+1}k!} u^k$$

into (3) quickly yields

$$f^{-1}(y) = \frac{1}{n} y^n \left[1 + \sum_{k=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \ldots (2k-1)}{2^k} \right) \left(\frac{n}{n + (2n-2)k} \right) \frac{(y^{2n-2})^k}{k!} \right].$$

So for any $n > 2$, the integral in (3) can be expressed as a Taylor series about $f = 0$ whose leading term is $(1/n)f^n$ and whose radius of convergence is 1. So x is a C^∞ function of f for $0 \leq f < 1$, and there are only two regions of an equizonal n-ovaloid which might have less than C^∞ smoothness: the left and right *ends* where the profile curve meets itself coming from n orthogonal directions and the *seam* where the two equizonal n-hemispheres comprising the equizonal n-ovaloid meet, which is a copy of $S^{n-1}(1)$.

At the ends, the projection onto the equizonal n-ovaloid of each of the n coordinate axes which is orthogonal to the x-axis looks like the curve $x = |f|^n$, which for even $n > 2$ is a C^∞ function of f and for odd $n > 2$ is a $C^{(n-1)/2}$ function of f. At the seam the smoothness of the equizonal n-ovaloid is determined by the smoothness of the *joint* in its profile curve, where the graph of f is reflected through the line $x = x_{\max} = f^{-1}(1)$.

We need to know how many derivatives the profile function f has at $x_{\max}$. Though we have no useful formula for f, this information can be teased out of (2). From (2) itself, it is clear that $\lim_{x \to x_{\max}^-} f'(x) = 0$, so the profile curve is at least C^1. Differentiating (2) yields

$$(n-1)\Big(1 + \big(f'(x)\big)^2\Big) + f(x)f''(x) = 0 \qquad (0 < x < x_{\max})$$

and differentiating again yields

$$(2n-1)f'(x)f''(x) + f(x)f'''(x) = 0 \qquad (0 < x < x_{\max})$$

from which respectively it follows that $\lim_{x \to x_{\max}^-} f''(x) = 1 - n$ and $\lim_{x \to x_{\max}^-} f'''(x) = 0$. This pattern holds up for higher derivatives. Repeated differentiations of (2) yield $S + f(x)f^{(m)}(x) = 0$ for $0 < x < x_{\max}$, where S is a sum of terms which look like $cf^{(i)}(x)f^{(j)}(x)$ for $0 < i, j < m$ such that when m is odd, every term of S has an odd derivative factor. Inductively it follows that f has left hand derivatives of all even orders at $x_{\max}$, and that its left hand derivatives of all odd orders are 0 at $x_{\max}$, so that the profile curve is C^∞ at the joint. $\square$

3. Mensuration formulas for the equizonal ovaloids

In order to mensurate the equizonal ovaloids, we need an exact value for the quantity

$$x_{\max} = f^{-1}(1) = \frac{1}{2}\int_0^1 \frac{t^{\frac{n-2}{2}}}{\sqrt{1-t^{n-1}}}\,\mathrm{d}t\,. \tag{4}$$

This can be obtained using the beta function:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t\,, \quad \mathrm{Re}(p) > 0 \quad \text{and} \quad \mathrm{Re}(q) > 0$$

via the identity

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\,. \tag{5}$$

Making the substitution $u = t^{n-1}$ in (4) and using (5) and the fact that $\Gamma(1/2) = \sqrt{\pi}$ yields

$$x_{\max} = \frac{1}{2n-2}\; B\left(\frac{n}{2n-2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{(2n-2)}\;\frac{\Gamma\left(\frac{n}{2n-2}\right)}{\Gamma\left(\frac{2n-1}{2n-2}\right)}\,.$$

Now it is easy to compute the n-volume of the unit equizonal n-ovaloid $EO^n(1)$ from the integral in (1) since the integrand is constant. Scaling the result by a factor of r^n yields a beautiful formula:

Proposition 2. *For $n \geq 2$, the n-volume of the equizonal n-ovaloid $EO^n(r)$ (whose largest $(n-1)$-dimensional spherical cross section has radius r) is*

$$\mathrm{vol}\big[EO^n(r)\big] = \frac{2\pi^{\frac{n+1}{2}}}{(n-1)}\;\frac{\Gamma\left(\frac{n}{2n-2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2n-1}{2n-2}\right)}\;r^n\,.$$

We can also compute the $(n+1)$-volume of the region in $\mathbb{R}^{n+1}$ enclosed by $EO^n(r)$, which we call an *equizonal* $(n+1)$*-ball* and denote $EB^{n+1}(r)$. Since the n-volume of $B^n(r)$ is given by

$$\mathrm{vol}\big[B^n(r)\big] = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}r^n\,, \quad n \geq 2$$

it follows that

$$\mathrm{vol}\big[EB^{n+1}(1)\big] = 2\int_0^{x_{\max}} \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}\big[f(x)\big]^n\;\mathrm{d}x\,.$$

We have no formula for f, but we can appeal to an old calculus trick: if $y = g(x)$ is a nonnegative, increasing function on $[0,b]$ and $g(0) = 0$, then $\int_0^b g(x)\;\mathrm{d}x = bg(b) - \int_0^{g(b)} g^{-1}(y)\;\mathrm{d}y$. Applying this to $g(x) = [f(x)]^n$ on $[0, x_{\max}]$ yields

$$\mathrm{vol}\big[EB^{n+1}(1)\big] = \frac{4\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}\left[x_{\max} - \int_0^1 f^{-1}\left(y^{\frac{1}{n}}\right)\;\mathrm{d}y\right].$$

Using the integral expression (3) for f^{-1} we have

$$\int_0^1 f^{-1}\left(y^{\frac{1}{n}}\right)\, \mathrm{d}y = \frac{1}{2}\int_0^1 \int_0^{y^{2/n}} \frac{t^{\frac{n-2}{2}}}{\sqrt{1-t^{n-1}}}\, \mathrm{d}t\, \mathrm{d}y$$

and interchanging the order of integration this can be written as

$$\frac{1}{2}\int_0^1 \int_{t^{n/2}}^1 \frac{t^{\frac{n-2}{2}}}{\sqrt{1-t^{n-1}}}\, \mathrm{d}y\, \mathrm{d}t$$

$$= \frac{1}{2}\int_0^1 \frac{(1-t^{\frac{n}{2}})t^{\frac{n-2}{2}}}{\sqrt{1-t^{n-1}}}\, \mathrm{d}t = x_{\max} - \frac{1}{2}\int_0^1 \frac{t^{n-1}}{\sqrt{1-t^{n-1}}}\, \mathrm{d}t\,.$$

The substitution $u = t^{n-1}$ converts this last integral into a beta function:

$$\frac{1}{2}\int_0^1 \frac{t^{n-1}}{\sqrt{1-t^{n-1}}}\, \mathrm{d}t = \frac{1}{2n-2}\, B\left(\frac{n}{n-1}, \frac{1}{2}\right) = \frac{1}{2n-2}\, \frac{\Gamma\left(\frac{n}{n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3n-1}{2n-2}\right)}$$

and yields another nice formula:

Proposition 3. *For $n \geq 2$, the $(n+1)$-volume of the equizonal $(n+1)$-ball $EB^{n+1}(r)$ enclosed by $EO^n(r)$ is*

$$\mathrm{vol}\big[EB^{n+1}(r)\big] = \frac{2\pi^{\frac{n+1}{2}}}{n(n-1)} \frac{\Gamma\left(\frac{n}{n-1}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{3n-1}{2n-2}\right)} r^{n+1}\,.$$

Of course when $n = 2$, the formulas in Propositions 2 and 3 reduce to $4\pi r^2$ and $4\pi r^3/3$, respectively.

4. Asymptotic behavior of EO^n and EB^{n+1} as $n \to \infty$

Using the above mensuration formulas, we observe the following curiosities.

The oblateness of $\mathbf{EO^n}$. Note that $\lim_{n\to\infty} x_{\max} = 0$. That is, in the limit as $n \to \infty$, $EO^n(1)$ becomes more and more oblate, looking more and more like two copies of the unit n-dimensional ball $B^n(1)$ glued together back to back in $(n+1)$-dimensional space. So one might expect vol$[EO^n(1)]$ to approach twice the n-volume of the unit n dimensional ball $B^n(1)$ as $n \to \infty$. But in fact

$$\frac{\mathrm{vol}[EO^n(1)]}{\mathrm{vol}[B^n(1)]} = \sqrt{\pi}\, \frac{n}{n-1}\, \frac{\Gamma\left(\frac{n}{2n-2}\right)}{\Gamma\left(\frac{2n-1}{2n-2}\right)} \to \pi \quad \text{as} \quad n \to \infty\,.$$

The n-volume of $\mathbf{EO^n(1)}$. Just as the n-volume of the unit n-dimensional sphere $S^n(1)$ increases as n increases from 1 to 6 and decreases monotonically to 0 as n increases from 6 to ∞, the n-volume of $EO^n(1)$ increases as n increases from 2 to 5 and decreases monotonically to 0 as n increases from 5 to ∞.

The $(\mathbf{n+1})$-volume of $\mathbf{EB^{n+1}(1)}$. Unlike the $(n+1)$-volume of the unit (n+1) dimensional ball $B^{n+1}(1)$ which increases as $n+1$ increases from 3 to 6 and decreases monotonically to 0 as $n+1$ increases from 6 to ∞, the $(n+1)$-volume of $EB^{n+1}(1)$ decreases monotonically to 0 as n increases from 2 to ∞.

The "surface area to volume" ratio for $\mathbf{EO^n}$. For equizonal ovaloids, the ratio $\mathrm{vol}[EO^n(1)]/\mathrm{vol}[EB^{n+1}(1)]$ behaves much like the corresponding ratio $\mathrm{vol}[S^n(1)]/\mathrm{vol}[B^{n+1}(1)]$ for spheres in that $\mathrm{vol}[S^n(1)]/\mathrm{vol}[B^{n+1}(1)] = n+1$ for $n \geq 2$ and $\mathrm{vol}[EO^n(1)]/\mathrm{vol}[EB^{n+1}(1)] = \frac{\pi}{2}n + o(n)$ as $n \to \infty$. So both ratios are asymptotically proportional to n as $n \to \infty$.

Archimedean ratios. Archimedes was particularly pleased to discover that if a sphere is inscribed in a cylinder, the ratio of the surface area of the sphere to the total surface area of the cylinder and the ratio of the volume enclosed by the sphere to the volume enclosed by the cylinder are both 2/3. It is easy to check that if the unit n-dimensional sphere $S^n(1)$ is inscribed in the n-dimensional hypercylinder $[0,2] \times S^{n-1}(1)$ then the ratio of the n-volume of $S^n(1)$ to the n-volume of the hypercylinder is the same as the ratio of the $(n+1)$-volume enclosed by $S^n(1)$ to the $(n+1)$ volume enclosed by the hypercylinder, namely

$$\frac{\mathrm{vol}[B^{(n+1)}(1)]}{2\ \mathrm{vol}[B^n(1)]} = \frac{\sqrt{\pi}}{2}\left(\frac{n}{n+1}\right)\frac{\Gamma(n/2)}{\Gamma((n+1)/2)}$$

which is 2/3 when $n=2$ but as $n \to \infty$ decreases monotonically towards a limit of 0.

In contrast, if the unit equizonal n-ovaloid $EO^n(1)$ is inscribed in the n-dimensional hypercylinder $[0, 2x_{\max}] \times S^{n-1}(1)$ then the ratio of the n-volume of $EO^n(1)$ to the n-volume of the hypercylinder is

$$\frac{\mathrm{vol}[EO^n(1)]}{2\ \mathrm{vol}[B^n(1)] + 2x_{\max}\ \mathrm{vol}[S^{n-1}(1)]} = \frac{1}{1 + \frac{2}{\sqrt{\pi}}\left(\frac{n-1}{n}\right)\Gamma\left(\frac{2n-1}{2n-2}\right) / \Gamma\left(\frac{n}{2n-2}\right)}$$

which is 2/3 when $n=2$ and as $n \to \infty$ decreases monotonically towards a limit of $\pi/(\pi+2) \approx ..611$, staying remarkably close to the classical Archimedean value of 2/3 for all values of n. Similarly, the ratio of the $(n+1)$-volume enclosed by $EO^n(1)$ to the $(n+1)$-volume enclosed by the circumscribed hypercylinder is

$$\frac{\mathrm{vol}[EB^{(n+1)}(1)]}{2x_{\max}\ \mathrm{vol}[B^n(1)]} = \frac{\Gamma\left(\frac{n}{n-1}\right)\Gamma\left(\frac{2n-1}{2n-2}\right)}{\Gamma\left(\frac{3n-1}{2n-2}\right)\Gamma\left(\frac{n}{2n-2}\right)}$$

which is 2/3 when $n=2$ and as $n \to \infty$ decreases monotonically towards a limit of $2/\pi \approx .637$. Though exact equality of these two ratios is not maintained for $n > 2$, both ratios stay remarkably close to 2/3 for all values of n.

5. Questions

Based on the some of the properties of the equizonal ovaloid that we have presented above, we would like to pose the following questions:

1. For odd $n \geq 3$: does there exist an n-dimensional closed, convex surface in $\mathbb{R}^{n+1}$ satisfying the generalized equal area zones property in one or more slicing directions and having C^∞ smoothness? (That is, are the singularities exhibited by EO^n for odd n an accident of our particular construction, or do they indicate a general phenomenon?)
2. For any $n \geq 3$, does there exist an n-dimensional closed, convex, C^∞ hypersurface in $\mathbb{R}^{n+1}$ satisfying the generalized equal area zones property in more than one slicing direction, or (best of all) in all slicing directions?

References

[1] H. T. Croft, K. J. Falconer, and R. K. Guy, Unsolved Problems in Geometry, Springer-Verlag, 1991.

[2] B. Richmond and T. Richmond, The equal area zones property, Amer. Math. Monthly **100** (1993) 475–477.

[3] O. Stamm, Umkehrung eines Satzes von Archimedes über die Kugel (German), Abh. Math. Sem. Univ. Hamburg **17** (1951) 112–132; reviewed in MathSciNet: MR0041467.

Jeff Dodd
Mathematical, Computing, and Information Sciences Department
Jacksonville State University
Jacksonville AL 36265
USA
e-mail: `jdodd@jsu.edu`

Vincent Coll
iConcepts, Inc.
331 North Broad St.
Lansdale PA 19446
USA
e-mail: `vecjr@iconcepts-inc.com`

Received: 2 July 2007.