Characterizing functions by hypervolumes revolution

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Prospectus In the CMJ, Richmond and Richmond [**1**] note that power functions $y = f(x) = x^\alpha$, $x \in [0, r]$, are characterized by a certain constant volume ratio associated with the surface of revolution generated by the graph of f. More specifically, for $r > 0$, let $R_1(r)$ be the first quadrant region under the curve $y = f(x)$ over an interval $[0, r]$, and let $R_2(r)$ be the first quadrant region to the left of $R1(r)$. Revolve these regions $R_1(r)$ and $R_2(r)$ around the y-axis to get solids of revolution with volumes $V_1(r)$ and $V_2(r)$, respectively. For $y = f(x) = x^\alpha$, there is the following characterizing ratio

$$\frac{V_1(r)}{V_2(r)} = \frac{\alpha}{2}.$$

We have extended this result to include hypersurfaces of revolution, so that in what follows $f(x)$ represents a profile curve for a hypersurface of revolution (in $\mathbb{R}^n$). In this case, the analogues of V_1 an V_2 may be computed via the disk method from calculus I except that the area of the disk must now be reckoned as the $(n-1)$-volume of an $(n-1)$ ball.

Adopting the notation from above we have the following theorem.

Theorem 1. *Suppose $f(x)$ is a continuous nonnegative function on $[0, b)$ satisfying the following conditions:*

1. $f'(x) > 0$ for $x \in (0, b)$,

2. $f''(x)$ *exists for* $x \in (0, b)$*, and*
3. the ratio $\frac{V_2(r)}{V_1(r)}$ *is the constant* $\frac{\alpha}{2}$ *for any* $r \in (0, b)$
then

$$f(x) = kx^{\frac{n}{\alpha-1}}$$

over $[0, b)$ *where* k *is some positive constant.*

Comments This project has introduced the student to higher dimensional volumes of revolution, the gamma function and basic techniques associated with solving ordinary differential equations.

References

[1] B. Richmond and T. Richmond, *Characterizing power functions by volumes of revolution*, College Math. Journal, **29** (Jan. 1998) 40-41.