A spherical ring is the object that remains when a cylindrical drill bit bores through a solid sphere along an axis, removing from the sphere a capsule consisting of a cylinder with a spherical cap on each end, as shown in Figure 1. Remarkably, the volume of such a spherical ring depends only on its height, defined as the height of its cylindrical inner boundary, and not on the radius of the sphere from which it was cut.

Figure 1: Cutting a spherical ring of height \( h \) from a sphere.

One straightforward way to verify this fact is to note that all the objects in Figure 1 are solids of revolution. This is depicted in Figure 2, where everything shown in the \( xy \)-plane is to be revolved around the \( x \)-axis. There
Figure 2: A spherical ring as a solid of revolution.

A sphere of radius \( r \) is represented by the semicircular graph of \( y = \sqrt{r^2 - x^2} \), and a spherical ring of height \( h \) cut from this sphere is represented by the shaded region below the semicircle and above the horizontal line segment of length \( h \) inscribed in the semicircle. We can calculate the volume of this spherical ring by integrating the areas of its annular cross-sections taken perpendicular to the \( x \)-axis (the “washer method”):

\[
V = \pi \int_{-h/2}^{h/2} \left[ (\sqrt{r^2 - x^2})^2 - \left( \sqrt{r^2 - (h/2)^2} \right)^2 \right] \, dx
\]

At the outset it looks as though \( V \) should depend on both \( r \) and \( h \), but it turns out to be a function of \( h \) only. This is a surprise that challenges many people’s intuition. For example, a spherical ring of height one centimeter cut out of a sphere the size of the earth has the same volume as a spherical ring of height one centimeter cut out of a sphere the size of a baseball. How can this be? The reason is that while the inner radius of the ring cut out of the earth is much larger, the radial thickness of this ring is much smaller: about \( 2 \times 10^{-10} \) cm, which is less than the diameter of a hydrogen atom. For spherical rings of any fixed height \( h \) cut out of spheres of increasing radius \( r \), this tradeoff between increasing inner radius (the quantity \( \sqrt{r^2 - (h/2)^2} \) in
Figure 2) and decreasing radial thickness (the quantity $r - \sqrt{r^2 - (h/2)^2}$ in Figure 2) preserves a fixed volume.

This property of the sphere appears in many calculus textbooks as an exercise in calculating volumes of solids of revolution. It has also caught the eye of many recreational mathematicians, perhaps getting its most public airing in the newspaper column of Marilyn vos Savant [11]. But, despite its prominence, it seems to lack a name. Since the process of cutting a spherical ring out of a sphere is much like coring an apple, we refer to this property as the coring property of the sphere.

Many surfaces of revolution can be similarly cored by cylindrical drill bits centered on their axes of revolution. So it is natural to ask to what extent the coring property characterizes the sphere among surfaces of revolution. Here we pose this question precisely and answer it completely using only elementary ideas from calculus, informed at critical junctures by geometric insight.

The coring property  The first order of business is to state the coring property in such a way that it applies to surfaces of revolution other than spheres. The coring property of the sphere compares spheres of different radii $r$, but each of these is just the unit sphere scaled up or down by the linear scale factor $r$. So we say that a surface of revolution satisfies the coring property if, when the surface is scaled up or down by a linear scale factor and then cored by a cylindrical drill bit centered on its axis of revolution, what remains (exterior to the drill bit) is a ring whose volume depends only on its height, and not on the scale factor. We define a ring to be a one-piece solid of revolution having a single cylindrical inner boundary, and the height of such a ring to be the height of its cylindrical inner boundary.

To flesh out this formulation of the coring property, and to give us a workable setup for our investigation of it, we need a picture. In general, a surface of revolution $S$ is generated by revolving a plane curve $C$, called the profile curve of $S$, around a line lying in the same plane as $C$, which
we have already called the *axis of revolution* of $S$. In particular, a sphere is the surface generated by revolving a semicircle around the line containing its diameter. (In fact, this is how Euclid defined a sphere in his *Elements* [6]!) Since we are essentially generalizing a property of the sphere, we begin with a profile curve looking much like a semicircle, as depicted in Figure 3.

![Figure 3: An even profile function $y = f(x)$ scaled by a linear scale factor $r$.](image)

The profile curve in Figure 3 is the graph of an even profile function $y = f(x)$ and is to be revolved around the $x$-axis. We scale the surface $S$ generated by the graph of $f$ by a linear scale factor $r$, yielding surfaces $S(r)$ generated by the curves $y/r = f(x/r)$, or $y = rf(x/r)$. (For example, if $S$ is a sphere of radius $\rho$, then $S(r)$ is a sphere of radius $r\rho$.) We can cut a ring out of the solid bounded by $S(r)$ by boring through it with a cylindrical drill bit centered on the $x$-axis. The resulting ring is generated by revolving the shaded region around the $x$-axis in Figure 3. We say that the surface $S$ satisfies the coring property if the volume $V(r, h)$ of a ring of height $h$ cut out of the solid bounded by $S(r)$ is a function of $h$ alone.

Before striking out in search of surfaces satisfying the coring property, let’s examine the assumptions implicit in Figure 3, since these will be the hypotheses for any conclusions that we reach based on this picture. To begin with, the profile curve in Figure 3 is not self-intersecting and it has exactly two $x$-intercepts. We accept these assumptions as geometrically natural,
because they ensure that the resulting surface $S$ is closed: that is, it encloses a single 3-dimensional region.

Two other prominent features of this profile curve are:

1. It is the graph of a function $y = f(x)$.

2. It has a vertical line of symmetry, which conveniently and with no loss of generality is the $y$-axis.

These assumptions are not quite as cumbersome as they might seem because, for our purposes, the first is subsumed by the second. That is, if a curve $C$ generates a surface that satisfies the coring property and if $C$ is symmetric with respect to the $y$-axis, then $y$ must be a function of $x$ on $C$. This is because for any profile curve $C$ that is symmetric with respect to the $y$-axis on which $y$ is not a function of $x$, there will be values of $h$ for which two or more rings having the same height $h$ but different volumes can be cut out of the surface generated by $C$ by cylindrical drill bits of different sizes, so that the volume of a ring cannot be a function of its height alone. For example, consider the profile curve $C$ indicated in Figure 4. For the value of $h$ indicated there, cylindrical drill bits of radii $R_1$, $R_2$, and $R_3$ will cut rings out of the surface generated by $C$ having the same height $h$ but

![Figure 4: A symmetric profile curve not defined by a function.](image)
different volumes. A surface generated by a curve $C$ having a vertical line of symmetry is centrally symmetric. That is, it has a center of symmetry: a point $P$ (in this case the origin) bisecting every line segment passing through $P$ that connects two points on the surface.

So a closed, centrally symmetric surface of revolution $S$ satisfying the coring property must be generated by the graph of an even profile function $f$ having exactly two $x$-intercepts. In addition, $f$ must be increasing to the left of $x = 0$ and decreasing to the right of $x = 0$, since only then will coring the surface $S$ with a cylindrical drill bit always result in what we have defined to be a ring, which needs to be in one piece. Therefore, to determine which closed, centrally symmetric surfaces of revolution satisfy the coring property, it is safe use Figure 3 as a starting point.

**The symmetric case: a calculus argument** The volume $V(r, h)$ of the ring formed in Figure 3 is twice the volume of the right half of the ring, which is the volume enclosed by $S(r)$ on the interval $0 \leq x \leq h/2$ less the volume of the cylinder drilled out on that same interval:

$$V(r, h) = 2 \left( \int_0^{h/2} \pi \left[ rf \left( \frac{x}{r} \right) \right]^2 \, dx - \pi \left[ rf \left( \frac{h}{2r} \right) \right]^2 \frac{h}{2} \right). \tag{1}$$

We wish to identify the functions $f$ for which $V$ depends only on $h$ and not on $r$. Towards this end, the simplest strategy turns out to be the best: we simply set equal to each other the volumes of two different rings of the same height, and see what we can say about $f$ based on the resulting equation.

In particular, note that for a ring cut out of the unscaled surface $S$, whose height $h$ will satisfy $0 \leq h/2 \leq a$, another ring of the same height can be cut out of any scaled-up surface $S(r)$ where $r > 1$, and the volumes of these two rings should be the same. That is, for any $h$ such that $0 \leq h/2 \leq a$ and any
\( r \geq 1 \), we should have \( V(1, h) = V(r, h) \), or from (1):

\[
2 \left[ \int_0^{h/2} \pi [f(x)]^2 \, dx - \pi \left[ f\left( \frac{h}{2} \right) \right]^2 \frac{h}{2} \right] = \\
2 \left[ \int_0^{h/2} \pi \left[ rf\left( \frac{x}{r} \right) \right]^2 \, dx - \pi \left[ rf\left( \frac{h}{2r} \right) \right]^2 \frac{h}{2} \right] 
\]

(2)

which is easily rearranged to yield

\[
\int_0^{h/2} ([f(x)]^2 - [rf(x/r)]^2) \, dx = (h/2) ([f(h/2)]^2 - r^2[f(h/2r)]^2). 
\]

(3)

For fixed \( r \geq 1 \), let

\[ g(x) = [f(x)]^2 - r^2[f(x/r)]^2. \]

Then for \( 0 \leq h/2 \leq a \), \( g \) satisfies

\[
\int_0^{h/2} g(x) \, dx = \frac{h}{2} g(h/2). 
\]

(4)

Dividing both sides of (4) by \( h/2 \), we see that the average value of \( g \) on any subinterval \([0, h/2]\) of \([0, a]\) is its value at the right endpoint of the subinterval: \( g(h/2) \). Does this mean that \( g \) must be constant? If \( f \) is continuous on the interval \([0, a]\), then so is \( g \), so that both sides of (4) are differentiable functions of \( h \). Differentiating yields

\[
\frac{1}{2} g(h/2) = \frac{1}{2} g(h/2) + \frac{h}{4} g'(h/2) 
\]

so that \( g'(h/2) = 0 \) for \( 0 \leq h/2 \leq a \). So indeed, \( g \) is constant on \([0, a]\). What is the constant? If, as in Figure 3, \( f(0) = b \), then

\[ g(0) = [f(0)]^2 - r^2[f(0)]^2 = b^2 - r^2b^2 = (1 - r^2)b^2 \]

so that

\[ [f(x)]^2 - r^2[f(x/r)]^2 = (1 - r^2)b^2. \]

(5)

If, as in Figure 3, \( f(a) = 0 \), then setting \( x = a \) in (5) yields

\[ [f(a/r)]^2 = \left( 1 - \frac{1}{r^2} \right) b^2. \]

(6)
This is essentially a formula for $f$. We can put it in a more recognizable form by making the change of variable $u = a/r$. Since $1 \leq r < \infty$, we have $0 < u \leq a$ and

$$[f(u)]^2 = \left(1 - \frac{u^2}{a^2}\right) b^2.$$  

That is, on the graph of $f$:

$$\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2 = 1. \tag{7}$$

So the graph of $f$ must be a semi-ellipse, which when revolved around the $x$-axis produces a spheroid: a sphere expanded or contracted in the $x$-direction. Indeed, direct calculation shows that the volume of a ring of height $h$ formed by coring the spheroid of equation (7) is

$$V_{\text{ring}} = \frac{1}{6} \pi \left(\frac{b}{a}\right)^2 h^3$$

which depends only on the shape of the spheroid and on $h$, and not on the scale of the spheroid. So we have shown:

**Proposition 1.** A closed, centrally symmetric surface of revolution generated by a continuous profile curve satisfies the coring property if and only if it is a spheroid.

The non-symmetric case: a geometric insight  
To expand our search for closed surfaces of revolution satisfying the coring property, we need to look at surfaces that are not centrally symmetric. But the profile curve of such a surface may not be the graph of a profile function. So how do we describe the profile curves among which we want to search? We must replace Figure 3 by the more complicated Figure 5.

There a non-symmetric profile curve $C$ generating a non-symmetric surface $S$ is scaled by a linear scale factor $r$ to produce a family of profile curves $C(r)$ that generate surfaces $S(r)$. For convenience, we locate the maximum $y$-value $b$ on the curve $C$ at the point $(0, b)$. Since by hypothesis the curve $C$ has exactly two $x$-intercepts, one portion of $C$ must connect the rightmost
of these $x$-intercepts with $(0, b)$ and another portion of $C$ must connect the leftmost of these $x$-intercepts with $(0, b)$. On each of these portions $y$ may not be a function of $x$, but $x$ is a function of $y$. Otherwise, coring the surface $S$ with a cylindrical drill bit centered on its axis would not always produce a ring, which by definition has to be in one piece. So the curve $C$ is the union of the graphs of two functions: $x = F(y)$ on the right and $x = G(y)$ on the left. The domain of both $F$ and $G$ is $0 \leq y \leq b$ and $F(b) = G(b) = 0$.

Fortunately, we can reduce this more complicated situation to the simpler one we have already analyzed. We merely symmetrize the profile curve $C$
in Figure 5 with respect to the $y$-axis. That is, for each $y$ we horizontally shift the line segment determined by the points $(G(y), y)$ and $(F(y), y)$ on $C$ so that its center is on the $y$-axis. The left and right endpoints of the shifted line segment then lie the same distance $(F(y) - G(y))/2$ to the left and the right of the $y$-axis, respectively. This transforms $C$ to the symmetric curve $C^*$ in Figure 6. The surface $S^*$ generated by $C^*$ is the symmetricization of the surface $S$ generated by $C$ relative to the plane $x = 0$. Clearly $S^*$ is centrally symmetric.

Now suppose we scale both the original curve $C$ and the symmetrized curve $C^*$ by the same linear scale factor $r$. Coring the resulting surfaces of revolution using the same cylindrical drill bit of radius $R$ centered on the $x$-axis yields two rings having the same height $h(r) = r(F(R/r) - G(R/r))$. These rings are generated by revolving the shaded regions around the $x$-axes in Figure 5 and Figure 6. If the volumes of these rings are calculated using the “shell method”, the answer is the same in each case:

$$V = \int_R^{rb} 2\pi y (rF(y/r) - rG(y/r)) \ dy.$$

It follows that the surface $S$ generated by the non-symmetric curve $C$ satisfies the coring property if and only if the centrally symmetric surface $S^*$ generated by the symmetrized curve $C^*$ does. If the curve $C$ is continuous (that is, if $F$ and $G$ are each continuous) then by Proposition 1, $S^*$ satisfies the coring property if and only if it is a spheroid. So we have shown:

**Proposition 2.** A closed surface of revolution generated by a continuous profile curve satisfies the coring property if and only if its symmetrization relative to a plane perpendicular to its axis of revolution is a spheroid.

**Examples** A variety of surfaces meet the hypotheses of Proposition 2 and therefore satisfy the coring property. The profile curve of each is the upper half of the graph of $(x/a)^2 + (y/b)^2 = 1$ “desymmetrized” by displacing each pair of points sharing a common $y$ value with a horizontal shift that varies continuously with $y$. For given positive $a$ and $b$, such profile curves can be
produced using either of the following recipes:

1. Choose a continuous “horizontal shift function” $h : [0, b] \rightarrow \mathbb{R}$, where $h(b) = 0$ to keep the maximum $y$-value on the curve at $(0, b)$. Then the profile curve is given by the upper half of the graph of

$$\left(\frac{(x - h(y))}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$ 

2. Choose a right hand portion for the curve: a continuous function $x = F(y)$ where $F : [0, b] \rightarrow \mathbb{R}$ and $F(b) = 0$, as in Figure 5. Then the left hand portion of the curve is given by $x = G(y) = F(y) - 2a\sqrt{1 - y^2/b^2}$.

Two profile curves created using the first recipe are shown in Figure 7 and Figure 8, and two created using the second recipe in Figure 9 and Figure 10. We have graphed the reflections of these profile curves through the $x$-axis as well, yielding “side views” of the resulting surfaces of revolution (which we have dubbed the egg, the Star Trek emblem, the acorn, and the heart respectively). In each case $a = b = 1$, so these curves all symmetrize to yield the unit sphere. We have not seen such non-symmetric examples exhibited elsewhere.

**Conclusions, Reflections, and Questions** Does the coring property characterize the sphere among closed surfaces of revolution? Based on Proposition 2, a fair answer is: “sort of”. Perhaps the largest class of surfaces that are at least vaguely sphere-like are *smooth ovaloids*: surfaces that are *convex*, meaning that the line segment connecting any two points inside the surface is also inside the surface, and *smooth*, meaning that near each point, the surface is the graph of a function having continuous partial derivatives of all orders, so that the surface has no sharp points or edges. Note that the surface in Figure 7 is a smooth ovaloid, but the surfaces generated by the profile curves in Figures 8 - 10 are, respectively, smooth but not convex, convex but not smooth, and neither smooth nor convex. The apparent diversity of these surfaces belies their unifying feature: they all yield spheroids when symmetrized.
Our investigation hardly exhausts the topic at hand. There are a number of lesser-known variations on the coring property of the sphere to be found in the literature. In his classic exploration of reasoning by induction and analogy *Mathematics and Plausible Reasoning* [8], George Polya noted that coring a sphere with conical or parabolic drill bits also produces rings whose volumes are determined by their heights alone. And Gerald Alexanderson has expanded on Polya’s observations by presenting an even larger catalog of similar phenomena [1].

This discussion may well bring to mind another interesting property of the sphere that can be found in the exercises of almost any calculus text: the fact that the surface area of a zone sliced out of a sphere by two parallel planes depends only on the distance between the planes and not on the location
of the zone. Does this “slicing property” characterize the sphere among closed surfaces of revolution? This question was addressed by B. Richmond and T. Richmond in the *Monthly* [9], where they named this property the *equal area zones property.* (The sphere turns out to be the only smooth surface of revolution satisfying this property, but some non-smooth surfaces of revolution satisfying this property can also be constructed.) More recently, a generalization of this property involving pairs of surfaces of revolution has been formulated and explored by Cass and Wildenberg [3]. Walter Rudin has formulated and examined a variation of the equal area zones property in the context of $n$-dimensional spheres [10]. And finally, we have examined higher dimensional analogs of both the equal area zones property [5] and the coring property [4] in the context of more general hypersurfaces of revolution.

Finally, here is a historical question. The machinery of calculus is not required to discover the coring property of the sphere. It can be derived elegantly using Cavalieri’s principle [7]. It can even be cobbled together from the volumes of a sphere, a cylinder, and a spherical cap, all of which were known to Archimedes [2]. Similarly, the equal area zones property of the sphere follows easily from a proposition of Archimedes (see [9]). But we know of no evidence that Archimedes noticed either of these properties. Moreover, it seems to us that it might have been difficult for him to have formulated them given the limitations of the language and notation of his day. Who was the the first to articulate these properties, and when?

**Acknowledgment.** We are grateful for the efforts of two anonymous referees who carefully read this article and made a number of suggestions that significantly improved its clarity and readability.

**References**


